$\mathcal{N}$-fold Parasupersymmetry

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Abstract

We find a new type of non-linear supersymmetries, called $\mathcal{N}$-fold parasupersymmetry, which is a generalization of both $\mathcal{N}$-fold supersymmetry and parasupersymmetry. We provide a general formulation of this new symmetry and then construct a second-order $\mathcal{N}$-fold parasupersymmetric quantum system where all the components of $\mathcal{N}$-fold parasupercharges are given by type A $\mathcal{N}$-fold supercharges. We show that this system exactly reduces to the Rubakov–Spiridonov model when $\mathcal{N} = 1$ and admits a generalized type C $2\mathcal{N}$-fold superalgebra. We conjecture the existence of other '$\mathcal{N}$-fold generalizations' such as $\mathcal{N}$-fold fractional supersymmetry, $\mathcal{N}$-fold orthosupersymmetry, and so on.

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In this letter, we report on a new type of non-linear supersymmetries, called $\mathcal{N}$-fold parasupersymmetry, which is a generalization of both $\mathcal{N}$-fold supersymmetry [1, 2, 3] and parasupersymmetry [4, 5, 6, 7, 8]. For the formulation, we fully employ the general formalism of parafermionic algebra and parasupersymmetry previously proposed by us in Ref. [9] and omit technical details of them in this letter. Hence, for the details see Ref. [9] and the references cited therein.

First of all, let us introduce parafermionic algebra of order $p(\in \mathbb{N})$. It is an associative algebra composed of the identity operator I and two parafermionic operators ψ^- and ψ^+ of order p which satisfy the nilpotency:

$$(\psi^-)^p \neq 0, \quad (\psi^+)^p \neq 0, \qquad (\psi^-)^{p+1} = (\psi^+)^{p+1} = 0. \tag{1}$$

Hence, we immediately have $2p+1$ non-zero elements $\{I, \psi^-, \dots, (\psi^-)^p, \psi^+, \dots, (\psi^+)^p\}$. We call them the *fundamental* elements of parafermionic algebra of order p. Parafermionic algebra is characterized by anti-commutation relation $\{A, B\} = AB + BA$ and commutation relation $[A, B] = AB - BA$ among the fundamental elements.

We shall next define parafermionic Fock spaces V_p of order p on which the parafermionic operators act. The latter space is $(p+1)$ dimensional and its $p+1$ bases $|k\rangle$ $(k = 0, \dots, p)$ are defined by

$$\psi^-|0\rangle = 0, \quad |k\rangle = (\psi^+)^k|0\rangle, \quad \psi^-|k\rangle = |k-1\rangle \quad (k = 1, \dots, p). \tag{2}$$

That is, ψ^- and ψ^+ act as annihilation and creation operators of parafermions, respectively. The state $|0\rangle$ is called the parafermionic *vacuum*. The subspace spanned by each state $|k\rangle$ $(k = 0, \dots, p)$ is called the k-parafermionic subspace and is denoted by $\mathsf{V}_p^{(k)}$. We can now define a set of projection operators $\Pi_k : \mathsf{V}_p \to \mathsf{V}_p^{(k)}$ $(k = 0, \dots, p)$ which satisfy

$$\Pi_k|l\rangle = \delta_{k,l}|k\rangle, \qquad \Pi_k\Pi_l = \delta_{k,l}\Pi_k, \qquad \sum_{k=0}^{p} \Pi_k = I. \tag{3}$$

From the definitions (2) and (3), we obtain

$$\Pi_{k+1}\psi^+ = \psi^+\Pi_k, \qquad \psi^-\Pi_{k+1} = \Pi_k\psi^-. \tag{4}$$

where and hereafter we put $\Pi_k \equiv 0$ for all $k < 0$ and $k > p$.

Parasupersymmetry of order 2 in quantum mechanics was first introduced by Rubakov and Spiridonov [4] and was later generalized to arbitrary order independently by Tomiya [6] and by Khare [7]. A different formulation for order 2 was proposed by Beckers and Debergh [5] and a generalization of the latter to arbitrary order was attempted by Chenaghlou and Fakhri [8]. Thus, we call them RSTK and BDCF formalism, respectively. We shall generalize them such that they reduce to $\mathcal{N}$-fold supersymmetry in Ref. [2] when parafermionic order is 1. For this purpose, we first introduce a pair of $\mathcal{N}$-fold parasupercharges $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$ of order p which satisfy

$$(\boldsymbol{Q}_{\mathcal{N}}^-)^p \neq 0, \quad (\boldsymbol{Q}_{\mathcal{N}}^+)^p \neq 0, \quad (\boldsymbol{Q}_{\mathcal{N}}^-)^{p+1} = (\boldsymbol{Q}_{\mathcal{N}}^+)^{p+1} = 0. \tag{5}$$

A system $\boldsymbol{H}$ is said to have *$\mathcal{N}$-fold parasupersymmetry of order p* if it commutes with the $\mathcal{N}$-fold parasupercharges of order p

$$[\boldsymbol{Q}_{\mathcal{N}}^{-}, \boldsymbol{H}] = [\boldsymbol{Q}_{\mathcal{N}}^{+}, \boldsymbol{H}] = 0, \tag{6}$$

and satisfies the non-linear relations in a generalized RSTK formalism

$$\sum_{k=0}^{p} (\boldsymbol{Q}_{\mathcal{N}}^{-})^{p-k} \boldsymbol{Q}_{\mathcal{N}}^{+} (\boldsymbol{Q}_{\mathcal{N}}^{-})^{k} = C_p (\boldsymbol{Q}_{\mathcal{N}}^{-})^{p-1} \mathsf{P}_{\mathcal{N}}(\boldsymbol{H}), \tag{7a}$$

$$\sum_{k=0}^{p} (\boldsymbol{Q}_{\mathcal{N}}^{+})^{p-k} \boldsymbol{Q}_{\mathcal{N}}^{-} (\boldsymbol{Q}_{\mathcal{N}}^{+})^{k} = C_p \mathsf{P}_{\mathcal{N}}(\boldsymbol{H}) (\boldsymbol{Q}_{\mathcal{N}}^{+})^{p-1}, \tag{7b}$$

where $\mathsf{P}_{\mathcal{N}}(x)$ is a monic polynomial of degree $\mathcal{N}$ in x, or in a generalized BDCF formalism

$$\underbrace{[\boldsymbol{Q}_{\mathcal{N}}^{-}, \cdots, [\boldsymbol{Q}_{\mathcal{N}}^{-},}_{(p-1)\ \text{times}} [\boldsymbol{Q}_{\mathcal{N}}^{+}, \boldsymbol{Q}_{\mathcal{N}}^{-}]] \cdots] = (-1)^p C_p (\boldsymbol{Q}_{\mathcal{N}}^{-})^{p-1} \mathsf{P}_{\mathcal{N}}(\boldsymbol{H}), \tag{8a}$$

$$\underbrace{[\boldsymbol{Q}_{\mathcal{N}}^{-}, \cdots, [\boldsymbol{Q}_{\mathcal{N}}^{+},}_{(p-1)\ \text{times}} [\boldsymbol{Q}_{\mathcal{N}}^{-}, \boldsymbol{Q}_{\mathcal{N}}^{+}]] \cdots] = C_p \mathsf{P}_{\mathcal{N}}(\boldsymbol{H}) (\boldsymbol{Q}_{\mathcal{N}}^{+})^{p-1}, \tag{8b}$$

where C_p is a constant. It is evident that in both of the generalizations (7) and (8) they reduce to ordinary parasupersymmetry of the corresponding formulations when $\mathcal{N} = 1$. As we pointed out in Ref. [9], an apparent drawback of the (generalized)BDCF formalism is that the relations (8) do not reduce to the ordinary $\mathcal{N}$-fold supersymmetric anti-commutation relation $\{\boldsymbol{Q}^{-}, \boldsymbol{Q}^{+}\} = C_1 \mathsf{P}_{\mathcal{N}}(\boldsymbol{H})$ when $p = 1$, in contrast to the RSTK relation (7). For this reason, we discard the (generalized) BDCF formalism in this paper though its defect may be amended by, e.g., replacing all the commutators in (8) by anti-commutators, graded commutators $[A, B\} = AB - (-1)^{\deg A \cdot \deg B} BA$, and so on.

An immediate consequence of the commutativity (6) is that each nth-power of the $\mathcal{N}$-fold parasupercharges $(2 \leq n \leq p)$ also commutes with the system $\boldsymbol{H}$

$$[(\boldsymbol{Q}_{\mathcal{N}}^{-})^n, \boldsymbol{H}] = [(\boldsymbol{Q}_{\mathcal{N}}^{+})^n, \boldsymbol{H}] = 0 \quad (2 \leq n \leq p). \tag{9}$$

Hence, every $\mathcal{N}$-fold parasupersymmetric system $\boldsymbol{H}$ satisfying (6) always has $2p$ conserved charges.

To realize $\mathcal{N}$-fold parasupersymmetry in quantum mechanical systems, we usually consider a vector space $\mathfrak{F} \times \mathsf{V}_p$ where $\mathfrak{F}$ is a linear space of complex functions such as the Hilbert space L^2 in Hermitian quantum theory and the Krein space $L^2_{\mathcal{P}}$ in $\mathcal{PT}$-symmetric quantum theory [10, 11]. An $\mathcal{N}$-fold parafermionic quantum system $\boldsymbol{H}$ is introduced by

$$\boldsymbol{H} = \sum_{k=0}^{p} H_k \Pi_k, \tag{10}$$

where H_k $(k = 0, \ldots, p)$ are scalar Hamiltonians acting on $\mathfrak{F}$:

$$H_k = -\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}q^2} + V_k(q) \quad (k = 0, \ldots, p). \tag{11}$$

Two $\mathcal{N}$-fold parasupercharges $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$ are defined by

$$\boldsymbol{Q}_{\mathcal{N}}^{-} = \sum_{k=0}^{p} Q_{\mathcal{N},k}^{-} \psi^{-} \Pi_k, \qquad \boldsymbol{Q}_{\mathcal{N}}^{+} = \sum_{k=0}^{p} Q_{\mathcal{N},k}^{+} \Pi_k \psi^{+}, \tag{12}$$

where $Q_{\mathcal{N},k}^{+}$ $(k = 0, \dots, p)$ are $\mathcal{N}$th-order linear operators acting on $\mathfrak{F}$

$$Q_{\mathcal{N},k}^{+} = \sum_{l=0}^{\mathcal{N}} w_{k,l}(q) \frac{\mathrm{d}^l}{\mathrm{d}q^l} \quad (k = 0, \dots, p), \tag{13}$$

and for each k $Q_{\mathcal{N},k}^{-}$ is given by a certain 'adjoint' of $Q_{\mathcal{N},k}^{+}$, e.g., the (ordinary) adjoint $Q_{\mathcal{N},k}^{-} = (Q_{\mathcal{N},k}^{+})^{\dagger}$ in the Hilbert space L^2, the $\mathcal{P}$-adjoint $Q_{\mathcal{N},k}^{-} = \mathcal{P}(Q_{\mathcal{N},k}^{+})^{\dagger}\mathcal{P}$ in the Krein space $L_{\mathcal{P}}^2$, and so on. For all $k \leq 0$ we put $Q_{\mathcal{N},k}^{\pm} \equiv 0$. When $p = 1$, the triple $(\boldsymbol{H}, \boldsymbol{Q}_{\mathcal{N}}^{-}, \boldsymbol{Q}_{\mathcal{N}}^{+})$ defined in Eqs. (10) and (12) becomes

$$\boldsymbol{H} = H_0 \psi^{-}\psi^{+} + H_1 \psi^{+}\psi^{-}, \quad \boldsymbol{Q}_{\mathcal{N}}^{-} = Q_{\mathcal{N},1}^{-}\psi^{-}, \quad \boldsymbol{Q}_{\mathcal{N}}^{+} = Q_{\mathcal{N},1}^{+}\psi^{+}, \tag{14}$$

and thus reduces to an ordinary $\mathcal{N}$-fold supersymmetric quantum mechanical system [2]. The non-linear relation (7) together with the nilpotency (5) for $p = 1$ are just the anti-commutation relations between supercharges

$$\{\boldsymbol{Q}_{\mathcal{N}}^{\pm}, \boldsymbol{Q}_{\mathcal{N}}^{\pm}\} = 0, \qquad \{\boldsymbol{Q}_{\mathcal{N}}^{-}, \boldsymbol{Q}_{\mathcal{N}}^{+}\} = C_1 \mathsf{P}_{\mathcal{N}}(\boldsymbol{H}). \tag{15}$$

Hence, the $\mathcal{N}$-fold parasupersymmetric quantum systems defined by Eqs. (5)–(13) provide a natural generalization of ordinary $\mathcal{N}$-fold supersymmetric quantum mechanics. It is easy to check that the $\mathcal{N}$-fold parasupercharges $\boldsymbol{Q}^{\pm}$ defined by Eq. (12) already satisfy the nilpotency (5) and that the commutativity (6) is satisfied if and only if

$$H_{k-1} Q_{\mathcal{N},k}^{-} = Q_{\mathcal{N},k}^{-} H_k, \quad Q_{\mathcal{N},k}^{+} H_{k-1} = H_k Q_{\mathcal{N},k}^{+}, \quad \forall k = 1, \dots, p. \tag{16}$$

That is, each pair of H_{k-1} and H_k must satisfy the intertwining relations with respect to the $\mathcal{N}$th-order linear differential operators $Q_{\mathcal{N},k}^{-}$ and $Q_{\mathcal{N},k}^{+}$. Similarly, the commutativity (9) between $(\boldsymbol{Q}_{\mathcal{N}}^{\pm})^n$ and $\boldsymbol{H}$ $(2 \leq n \leq p)$ means that any pair of H_{k-n} and H_k $(1 \leq n \leq k \leq p)$ satisfies

$$H_{k-n} Q_{\mathcal{N},k-n+1}^{-} \cdots Q_{\mathcal{N},k-1}^{-} Q_{\mathcal{N},k}^{-} = Q_{\mathcal{N},k-n+1}^{-} \dots Q_{\mathcal{N},k-1}^{-} Q_{\mathcal{N},k}^{-} H_k, \tag{17a}$$

$$Q_{\mathcal{N},k}^{+} Q_{\mathcal{N},k-1}^{+} \dots Q_{\mathcal{N},k-n+1}^{+} H_{k-n} = H_k Q_{\mathcal{N},k}^{+} Q_{\mathcal{N},k-1}^{+} \cdots Q_{\mathcal{N},k-n+1}^{+}, \tag{17b}$$

which means that H_{k-n} and H_k constitute a pair of $n\mathcal{N}$-fold supersymmetry. The relations (17) can be also derived by repeated applications of Eq. (16). Since $\mathcal{N}$-fold supersymmetry is essentially equivalent to weak quasi-solvability [2, 12], $\mathcal{N}$-fold parasupersymmetric quantum systems also possess weak quasi-solvability. To see the structure of weak quasi-solvability in the $\mathcal{N}$-fold parasupersymmetric system $\boldsymbol{H}$ more precisely, let us first define

$$\mathcal{V}_{n,k}^{-} = \ker(Q_{\mathcal{N},k-n+1}^{-} \cdots Q_{\mathcal{N},k}^{-}), \quad \mathcal{V}_{n,k}^{+} = \ker(Q_{\mathcal{N},k}^{+} \cdots Q_{\mathcal{N},k-n+1}^{+}) \quad (1 \leq n \leq k \leq p). \tag{18}$$

By the definition (18), the vector spaces $\mathcal{V}_{n,k}^{\pm}$ for each fixed k are related as

$$\mathcal{V}_{1,k}^{-} \subset \mathcal{V}_{2,k}^{-} \subset \cdots \subset \mathcal{V}_{k,k}^{-}, \quad \mathcal{V}_{1,k}^{+} \subset \mathcal{V}_{2,k}^{+} \subset \cdots \subset \mathcal{V}_{k,k}^{+}, \tag{19}$$

On the other hand, it is evident from the intertwining relations (17) that each Hamiltonian H_k $(0 \leq k \leq p)$ preserves vector spaces as follows:

$$H_k \mathcal{V}_{n,k}^- \subset \mathcal{V}_{n,k}^- \quad (1 \leq n \leq k), \tag{20a}$$

$$H_k \mathcal{V}_{n,k+n}^+ \subset \mathcal{V}_{n,k+n}^+ \quad (1 \leq n \leq p-k). \tag{20b}$$

From Eqs. (19) and (20), the largest space preserved by each H_k $(0 \leq k \leq p)$ is given by

$$\mathcal{V}_{k,k}^- + \mathcal{V}_{p-k,p}^+ \quad (0 \leq k \leq p). \tag{21}$$

Needless to say, each Hamiltonian H_k preserves the two spaces in Eq. (21) separately. The intertwining relations (16) and (17) ensure that all the component Hamiltonians H_k $(k = 0, \dots, p)$ of the system $\boldsymbol{H}$ are isospectral outside the sectors $\mathcal{V}_{n,k}^\pm$ $(1 \leq n \leq k \leq p)$. The spectral degeneracy of $\boldsymbol{H}$ in these sectors depends on the form of each component of the $\mathcal{N}$-fold parasupercharges, $Q_{\mathcal{N},k}^\pm$ $(k = 1, \dots, p)$.

In addition to these 'power-type' symmetries, every $\mathcal{N}$-fold parasupersymmetric quantum system $\boldsymbol{H}$ defined in Eq. (10) can have 'discrete-type' ones. The conserved charges of this type are given by

$$\boldsymbol{Q}_{\mathcal{N},\{n\}}^\pm = [\{(\psi^-)^n, (\psi^+)^n\}, \boldsymbol{Q}_{\mathcal{N}}^\pm], \quad \boldsymbol{Q}_{\mathcal{N},[n]}^\pm = [[(\psi^-)^n, (\psi^+)^n], \boldsymbol{Q}_{\mathcal{N}}^\pm] \quad (n = 1, \dots, p), \tag{22}$$

It follows from Jacobi identity that they indeed commute with $\boldsymbol{H}$:

$$[\boldsymbol{Q}_{\mathcal{N},\{n\}}^\pm, \boldsymbol{H}] = [\boldsymbol{Q}_{\mathcal{N},[n]}^\pm, \boldsymbol{H}] = 0 \quad (n = 1, \dots, p). \tag{23}$$

We note, however, that they are in general not linearly independent and we cannot determine the number of linearly independent conserved charges without the knowledge of parafermionic algebra of each order.

The non-linear constraints (7) can be also calculated in a similar way. The first non-linear relation in Eq. (7) is satisfied if and only if the following two identities hold:

$$Q_{\mathcal{N},1}^- \cdots Q_{\mathcal{N},p}^- Q_{\mathcal{N},p}^+ + \sum_{k=1}^{p-1} Q_{\mathcal{N},1}^- \cdots Q_{\mathcal{N},p-k}^- Q_{\mathcal{N},p-k}^+ Q_{\mathcal{N},p-k}^- \cdots Q_{\mathcal{N},p-1}^-$$
$$= C_p Q_{\mathcal{N},1}^- \cdots Q_{\mathcal{N},p-1}^- \mathsf{P}_{\mathcal{N}}(H_{p-1}), \tag{24a}$$

$$\sum_{k=1}^{p-1} Q_{\mathcal{N},2}^- \cdots Q_{\mathcal{N},p-k+1}^- Q_{\mathcal{N},p-k+1}^+ Q_{\mathcal{N},p-k+1}^- \cdots Q_{\mathcal{N},p}^- + Q_{\mathcal{N},1}^+ Q_{\mathcal{N},1}^- \cdots Q_{\mathcal{N},p}^-$$
$$= C_p Q_{\mathcal{N},2}^- \cdots Q_{\mathcal{N},p}^- \mathsf{P}_{\mathcal{N}}(H_p). \tag{24b}$$

The conditions for the second non-linear relation in Eq. (7) are apparently given by the 'adjoint' of Eqs. (24).

An '$\mathcal{N}$-fold generalization' of quasi-parasupersymmetry proposed in Ref. [9] is also straightforward.

Let us now construct a second-order $\mathcal{N}$-fold parasupersymmetric quantum system. In the case of $p = 2$, the triple $(\boldsymbol{H}, \boldsymbol{Q}_{\mathcal{N}}^-, \boldsymbol{Q}_{\mathcal{N}}^+)$ defined in Eqs. (10) and (12) is given by

$$\boldsymbol{H} = H_0(\psi^-)^2(\psi^+)^2 + H_1(\psi^+\psi^- - (\psi^+)^2(\psi^-)^2) + H_2(\psi^+)^2(\psi^-)^2, \tag{25}$$

$$\boldsymbol{Q}_{\mathcal{N}}^- = Q_{\mathcal{N},1}^-(\psi^-)^2\psi^+ + Q_{\mathcal{N},2}^-\psi^+(\psi^-)^2, \tag{26}$$

$$\boldsymbol{Q}_{\mathcal{N}}^+ = Q_{\mathcal{N},1}^+\psi^-(\psi^+)^2 + Q_{\mathcal{N},2}^+(\psi^+)^2\psi^-. \tag{27}$$

We recall the fact that the above second-order parasupercharges (26) and (27) already satisfy the nilpotent condition (5) for $p = 2$, $(\boldsymbol{Q}_{\mathcal{N}}^{-})^3 = (\boldsymbol{Q}_{\mathcal{N}}^{+})^3 = 0$. From Eqs. (16) and (24), the commutativity (6) and the non-linear constraints (7) for $p = 2$ hold if and only if the following conditions

$$H_0 Q_{\mathcal{N},1}^{-} = Q_{\mathcal{N},1}^{-} H_1, \qquad H_1 Q_{\mathcal{N},2}^{-} = Q_{\mathcal{N},2}^{-} H_2, \tag{28}$$

$$Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},2}^{-} Q_{\mathcal{N},2}^{+} + Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},1}^{+} Q_{\mathcal{N},1}^{-} = C_2 Q_{\mathcal{N},1}^{-} \mathsf{P}_{\mathcal{N}}(H_1), \tag{29}$$

$$Q_{\mathcal{N},2}^{-} Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},2}^{-} + Q_{\mathcal{N},1}^{+} Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},2}^{-} = C_2 Q_{\mathcal{N},2}^{-} \mathsf{P}_{\mathcal{N}}(H_2), \tag{30}$$

and their 'adjoint' relations

$$Q_{\mathcal{N},1}^{+} H_0 = H_1 Q_{\mathcal{N},1}^{+}, \qquad Q_{\mathcal{N},2}^{+} H_1 = H_2 Q_{\mathcal{N},2}^{+}, \tag{31}$$

$$Q_{\mathcal{N},1}^{+} Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},1}^{+} + Q_{\mathcal{N},2}^{-} Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},1}^{+} = C_2 \mathsf{P}_{\mathcal{N}}(H_1) Q_{\mathcal{N},1}^{+}, \tag{32}$$

$$Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},1}^{+} Q_{\mathcal{N},1}^{-} + Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},2}^{-} Q_{\mathcal{N},2}^{+} = C_2 \mathsf{P}_{\mathcal{N}}(H_2) Q_{\mathcal{N},2}^{+}, \tag{33}$$

are satisfied. In general, we do not need to solve the 'adjoint' conditions.

For the second-order case, we have one new $\mathcal{N}$-fold quasi-parasupersymmetry, namely, that of order $(2,2)$. The conditions are given by Eqs. (28)–(33) but the first-order intertwining relations (28) and (31) are replaced by the second-order intertwining relations

$$H_0 Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},2}^{-} = Q_{\mathcal{N},1}^{-} Q_{\mathcal{N},2}^{-} H_2, \qquad Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},1}^{+} H_0 = H_2 Q_{\mathcal{N},2}^{+} Q_{\mathcal{N},1}^{+}. \tag{34}$$

Let us next put $C_2 = 2^{\mathcal{N}+1}$ and

$$H_k = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V_k(q), \qquad Q_{\mathcal{N},k}^{+} = \prod_{i=0}^{\mathcal{N}-1}\left(\frac{\mathrm{d}}{\mathrm{d}q} + W_k(q) + \frac{\mathcal{N}-1-2i}{2}E_k(q)\right), \tag{35}$$

where the product of operators is ordered according to $\prod_{i=i_0}^{i_1} A_i = A_{i_1} A_{i_1 - 1} \cdots A_{i_0}$. Each component $Q_{\mathcal{N},k}^{+}$ of $\mathcal{N}$-fold parasupercharges given in Eq. (35) is so-called type A $\mathcal{N}$-fold supercharge, and the necessary and sufficient condition for two Hamiltonians to be intertwined by it is already well known [12]; the conditions (28) and (31) are satisfied if and only if

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{1}{2}W_1(q)^2 + \frac{\mathcal{N}^2-1}{24}\big(E_1(q)^2 - 2E_1'(q)\big) - \frac{\mathcal{N}}{2}W_1'(q) - R_1, \tag{36}$$

$$\begin{aligned} H_1 &= -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{1}{2}W_1(q)^2 + \frac{\mathcal{N}^2-1}{24}\big(E_1(q)^2 - 2E_1'(q)\big) + \frac{\mathcal{N}}{2}W_1'(q) - R_1 \\ &= -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{1}{2}W_2(q)^2 + \frac{\mathcal{N}^2-1}{24}\big(E_2(q)^2 - 2E_2'(q)\big) - \frac{\mathcal{N}}{2}W_2'(q) - R_2, \end{aligned} \tag{37}$$

$$H_2 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{1}{2}W_2(q)^2 + \frac{\mathcal{N}^2-1}{24}\big(E_2(q)^2 - 2E_2'(q)\big) + \frac{\mathcal{N}}{2}W_2'(q) - R_2, \tag{38}$$

where R_k $(k = 1, 2)$ are constants and the functions E_k and W_k $(k = 1, 2)$ must satisfy the following non-linear differential equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}q} - E_k(q)\right)\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}}{\mathrm{d}q} + E_k(q)\right)W_k(q) = 0 \quad \text{for} \quad \mathcal{N} \geq 2, \tag{39}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}q} - 2E_k(q)\right)\left(\frac{\mathrm{d}}{\mathrm{d}q} - E_k(q)\right)\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\mathrm{d}}{\mathrm{d}q} + E_k(q)\right)E_k(q) = 0 \quad \text{for} \quad \mathcal{N} \geq 3. \tag{40}$$

We note that the formula (37) for H_1 implies the following condition among E_k and W_k:

$$W_1^2 + \frac{\mathcal{N}^2 - 1}{12}\left(E_1^2 - 2E_1'\right) + \mathcal{N}W_1' - 2R_1 = W_2^2 + \frac{\mathcal{N}^2 - 1}{12}\left(E_2^2 - 2E_2'\right) - \mathcal{N}W_2' - 2R_2. \tag{41}$$

It is worth pointing out that it is similar to but less restrictive than the condition for simultaneous type A $\mathcal{N}$-fold supersymmetry with two different values of $\mathcal{N}$, cf. Eqs. (15) and (16) in Ref. [13]. When the conditions (36)–(40) are all satisfied, it was shown [12] that the following relations hold

$$Q_{\mathcal{N},1}^{-}Q_{\mathcal{N},1}^{+} = 2^{\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_0), \qquad Q_{\mathcal{N},1}^{+}Q_{\mathcal{N},1}^{-} = 2^{\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_1), \tag{42a}$$

$$Q_{\mathcal{N},2}^{-}Q_{\mathcal{N},2}^{+} = 2^{\mathcal{N}}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_1), \qquad Q_{\mathcal{N},2}^{+}Q_{\mathcal{N},2}^{-} = 2^{\mathcal{N}}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_2), \tag{42b}$$

where $\pi_{k,\mathcal{N}}^{[\mathcal{N}]}$ are the $\mathcal{N}$th critical generalized Bender–Dunne polynomials associated with each system labeled by the indices $k = 1, 2$. Substituting Eqs. (42) into the second condition (29), we have

$$2^{\mathcal{N}}Q_{\mathcal{N},1}^{-}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_1) + 2^{\mathcal{N}}Q_{\mathcal{N},1}^{-}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_1) = C_2 Q_{\mathcal{N},1}^{-}\mathsf{P}_{\mathcal{N}}(H_1), \tag{43}$$

and thus obtain a solution to the condition (29) as

$$\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(x) + \pi_{2,\mathcal{N}}^{[\mathcal{N}]}(x) = 2\mathsf{P}_{\mathcal{N}}(x). \tag{44}$$

Finally, substituting Eqs. (42) and (44) into the third condition (30), we have

$$\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_1)Q_{\mathcal{N},2}^{-} + \pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_1)Q_{\mathcal{N},2}^{-} = 2\mathsf{P}_{\mathcal{N}}(H_1)Q_{\mathcal{N},2}^{-} = 2Q_{\mathcal{N},2}^{-}\mathsf{P}_{\mathcal{N}}(H_2). \tag{45}$$

It is evident that this condition is already satisfied since we have constructed the system so that H_1 and H_2 satisfy the second intertwining relation in Eq. (28). Therefore, the system (35)–(38) constitutes a second-order $\mathcal{N}$-fold parasupersymmetric quantum system with the monic $\mathcal{N}$th-degree polynomial $\mathsf{P}_{\mathcal{N}}$ given by Eq. (44) provided that the conditions (39)–(41) are all satisfied. We note that this system exactly reduces to the parasupersymmetric quantum system of Rubakov–Spiridonov (RS) type [4] when $\mathcal{N} = 1$ and $R_1 + R_2 = 0$.

In our previous paper [9], we found that the RS model admits a generalized 2-fold superalgebra. In the following, we show that the above $\mathcal{N}$-fold parasupersymmetric system also satisfies a novel non-linear algebra. Using the relation (42) and applying the intertwining relation (31), we obtain for the system (35)–(38) the following formulas:

$$\boldsymbol{Q}_{\mathcal{N}}^{-}\boldsymbol{Q}_{\mathcal{N}}^{+} = 2^{\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_0)\Pi_0 + 2^{\mathcal{N}}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_1)\Pi_1, \tag{46}$$

$$\boldsymbol{Q}_{\mathcal{N}}^{+}\boldsymbol{Q}_{\mathcal{N}}^{-} = 2^{\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_1)\Pi_1 + 2^{\mathcal{N}}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_2)\Pi_2, \tag{47}$$

$$\begin{aligned}(\boldsymbol{Q}_{\mathcal{N}}^{-})^2(\boldsymbol{Q}_{\mathcal{N}}^{+})^2 &= 2^{\mathcal{N}}Q_{\mathcal{N},1}^{-}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_1)Q_{\mathcal{N},1}^{+}\Pi_0 = 2^{\mathcal{N}}Q_{\mathcal{N},1}^{-}Q_{\mathcal{N},1}^{+}\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_0)\Pi_0\\ &= 2^{2\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_0)\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_0)\Pi_0,\end{aligned} \tag{48}$$

$$\begin{aligned}(\boldsymbol{Q}_{\mathcal{N}}^{+})^2(\boldsymbol{Q}_{\mathcal{N}}^{-})^2 &= 2^{\mathcal{N}}Q_{\mathcal{N},2}^{+}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_1)Q_{\mathcal{N},2}^{-}\Pi_2 = 2^{\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_2)Q_{\mathcal{N},2}^{+}Q_{\mathcal{N},2}^{-}\Pi_2\\ &= 2^{2\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(H_2)\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(H_2)\Pi_2.\end{aligned} \tag{49}$$

Hence, we can easily find the following non-linear relation:

$$(\boldsymbol{Q}_{\mathcal{N}}^{-})^2(\boldsymbol{Q}_{\mathcal{N}}^{+})^2+\boldsymbol{Q}_{\mathcal{N}}^{\pm}(\boldsymbol{Q}_{\mathcal{N}}^{\mp})^2\boldsymbol{Q}_{\mathcal{N}}^{\pm}+(\boldsymbol{Q}_{\mathcal{N}}^{+})^2(\boldsymbol{Q}_{\mathcal{N}}^{-})^2=2^{2\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(\boldsymbol{H})\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(\boldsymbol{H}). \tag{50}$$

It is interesting to note that this non-linear relation can be regarded as a generalization of $2\mathcal{N}$-fold superalgebra. Indeed, if we restrict the linear space $\mathfrak{F}\times\mathsf{V}_2$ on which the system $\boldsymbol{H}$ acts to $\mathfrak{F}\times(\mathsf{V}_2^{(0)}\dotplus\mathsf{V}_2^{(2)})$ (cf. the definition between Eqs. (2) and (3)), we have

$$\{(\boldsymbol{Q}_{\mathcal{N}}^{-})^2,(\boldsymbol{Q}_{\mathcal{N}}^{+})^2\}=2^{2\mathcal{N}}\pi_{1,\mathcal{N}}^{[\mathcal{N}]}(\boldsymbol{H})\pi_{2,\mathcal{N}}^{[\mathcal{N}]}(\boldsymbol{H})\big|_{\mathfrak{F}\times(\mathsf{V}_2^{(0)}\dotplus\mathsf{V}_2^{(2)})}. \tag{51}$$

This, together with the trivial (anti-)commutation relations

$$\{(\boldsymbol{Q}_{\mathcal{N}}^{-})^2,(\boldsymbol{Q}_{\mathcal{N}}^{-})^2\}=\{(\boldsymbol{Q}_{\mathcal{N}}^{+})^2,(\boldsymbol{Q}_{\mathcal{N}}^{+})^2\}=[(\boldsymbol{Q}_{\mathcal{N}}^{\pm})^2,\boldsymbol{H}]=0, \tag{52}$$

constitutes a type of $2\mathcal{N}$-fold superalgebra in the sector $\mathfrak{F}\times(\mathsf{V}_2^{(0)}\dotplus\mathsf{V}_2^{(2)})$. We also note that the anti-commutation relation (51) is reminiscent of the one appeared in type C $\mathcal{N}$-fold supersymmetry, cf. Eq. (5.11b) in Ref. [14]. It is not accidental. Indeed, on one hand it follows from Eqs. (28) and (31) that H_0 and H_2 are intertwined by $Q_{\mathcal{N},2}^{+}Q_{\mathcal{N},1}^{+}$, which is the component of $(\boldsymbol{Q}_{\mathcal{N}}^{+})^2$, and on the other hand if we put

$$E_1=E_2=E,\quad W_1-\frac{\mathcal{N}}{2}E_1=W,\quad W_2+\frac{\mathcal{N}}{2}E_2=W+(\mathcal{N}-\lambda)F, \tag{53}$$

where λ is a parameter, the operator $Q_{\mathcal{N},2}^{+}Q_{\mathcal{N},1}^{+}$ is expressed as

$$\begin{aligned}Q_{\mathcal{N},2}^{+}Q_{\mathcal{N},1}^{+}=&\prod_{i=\mathcal{N}}^{2\mathcal{N}-1}\left(\frac{\mathrm{d}}{\mathrm{d}q}+W+(\mathcal{N}-\lambda)F+\frac{2\mathcal{N}-1-2i}{2}E\right)\\&\times\prod_{i=0}^{\mathcal{N}-1}\left(\frac{\mathrm{d}}{\mathrm{d}q}+W+\frac{2\mathcal{N}-1-2i}{2}E\right),\end{aligned} \tag{54}$$

which is nothing but a type C $2\mathcal{N}$-fold supercharge with $\mathcal{N}_1=\mathcal{N}_2=\mathcal{N}$ (cf. Eq. (3.22) in Ref. [14]). Hence, H_0 and H_2 can be regarded as a type C $2\mathcal{N}$-fold supersymmetric pair and thus the formula (51) can be naturally understood.

We also note that as in the case of (ordinary) parasupersymmetry, quasi-parasupersymmetry of order $(2,2)$ may not produce any new result over parasupersymmetry of order 2. The reason is that in most cases the general solution to the conditions (29) and (32) would be given by

$$Q_{\mathcal{N},1}^{+}Q_{\mathcal{N},1}^{-}+Q_{\mathcal{N},2}^{-}Q_{\mathcal{N},2}^{+}=C_2\mathsf{P}_{\mathcal{N}}(H_1). \tag{55}$$

In this case, the conditions (30) and (33) are equivalent to

$$\mathsf{P}_{\mathcal{N}}(H_1)Q_{\mathcal{N},2}^{-}=Q_{\mathcal{N},2}^{-}\mathsf{P}_{\mathcal{N}}(H_2),\qquad Q_{\mathcal{N},2}^{+}\mathsf{P}_{\mathcal{N}}(H_1)=\mathsf{P}_{\mathcal{N}}(H_2)Q_{\mathcal{N},2}^{+}, \tag{56}$$

which is close to the second relation in Eqs. (28) and (31). Hence, in most of second-order cases, $\mathcal{N}$-fold quasi-parasupersymmetry may be identical to $\mathcal{N}$-fold parasupersymmetry.

Finally, we would like to refer to the fact that some different types of generalized supersymmetries have been shown to have intimate relation with each other. For instance, according to Ref. [15] every orthosupersymmetric systems [16] admits both parasupersymmetry and fractional supersymmetry [17]. Combining this fact with the present results shown in this letter, we conjecture the existence of '$\mathcal{N}$-fold generalization' of other supersymmetric variants such as *$\mathcal{N}$-fold fractional supersymmetry* characterized by the following non-linear relation

$$(\boldsymbol{Q}_{\mathcal{N},i})^{p} = C_{p}\mathsf{P}_{\mathcal{N}}(\boldsymbol{H}), \tag{57}$$

$\mathcal{N}$-fold orthosupersymmetry characterized by the following non-linear relation

$$\boldsymbol{Q}_{\mathcal{N},\alpha}^{\pm}\boldsymbol{Q}_{\mathcal{N},\beta}^{\pm} = 0, \qquad \boldsymbol{Q}_{\mathcal{N},\alpha}^{-}\boldsymbol{Q}_{\mathcal{N},\beta}^{+} + \delta_{\alpha,\beta}\sum_{\gamma=1}^{p}\boldsymbol{Q}_{\mathcal{N},\gamma}^{+}\boldsymbol{Q}_{\mathcal{N},\gamma}^{-} = C_{p}\delta_{\alpha,\beta}\mathsf{P}_{\mathcal{N}}(\boldsymbol{H}), \tag{58}$$

and so on.

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